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Propagation in the earth-ionosphere waveguide by the multi-scaling method

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SUMMARY
A problem of propagation of modulated high-frequency signals in a waveguide of nonconstant height and nonconstant wall impedance is analysed. A multiple-scale method is used, which reduces the three-dimensional problem to a two-dimensional problem for each mode. The modal propagation problem is analysed by the ray method, giving rise to phase and group velocities, and showing the possibility of focusing effects due to variation in height and wall impedance. Mode coupling effects are shown and mode coupling coefficients calculated.

1. Introduction

An interesting problem of guided waves is that of propagation of various natural and man-made signals in the earth-ionosphere waveguide. Some of the factors that make this problem mathematically difficult are: (a) the earth ionosphere waveguide is three-dimensional, (b) its height is variable, (c) the earth conductivity (or impedance) is variable, (d) many of the signals are not time-harmonic. In this paper we shall demonstrate an asymptotic method which yields physically meaningful approximate solutions to a problem characterized by the above-mentioned properties (a)–(d). The method assumes that the waveguide is thin (compared to lateral dimensions). It is useful in the range in which guided mode representations are useful, namely: when the guides' height is of the same order as the wavelength of the carrier frequency. We assume a flat (rather than a spherical) geometry, and analyse the behavior of a scalar field (rather than electromagnetic field vectors) in order to keep the calculations simple. However, this method depends only on proper scaling, which results from geometric considerations, and is not tailored to the equations and boundary conditions at hand. Thus it can be applied to a large class of guided wave propagation problem [1], [2], [3]. Actually, we assume no a priori knowledge of the solution except the scaling (2.4) and (3.1) and derive everything from there. This is an advantage of our approach over some previous work [4], [5]. The method reduces a propagation problem in a three dimensional domain which is "thin", but otherwise quite general, to a problem of propagation in an infinite two-dimensional domain for each mode. The two dimensional problem for the modes can be solved by ray methods. In the lowest approximation the modes are uncoupled. But higher approximations reveal mode coupling, which we show how to calculate. This is another advantage of this approach over previous methods [4], [5].

2. Formulation of the problem

Let $V(r, \theta, z, t)$ be a scalar potential function which satisfies the wave equation

$$c^2 \Delta_3 V - V_{tt} \equiv c^2 \left(V_{rr} + \frac{1}{r} V_r + \frac{1}{r^2} V_{\theta\theta} + V_{zz} \right) - V_{tt} = 0 \tag{2.1}$$

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for $t>0$, in a waveguide region D, given by

$$r_0(\theta) \leqq r < \infty, \quad 0 \leqq \theta \leqq 2\pi, \quad 0 \leqq z \leqq \varepsilon h(r, \theta). \tag{2.2}$$

The function $h(r, \theta)$ is bounded, positive and "smooth enough", $r_0(\theta)$ is some closed curve in the r, θ plane and ε is a small parameter, expressing the fact that the guide is thin. The boundary and initial conditions we choose are

$$V_z + \varepsilon^{-1} X(r, \theta) V = 0, \quad \text{at} \quad z = 0, \tag{2.3a}$$
$$V = 0, \quad \text{at} \quad z = \varepsilon h(r, \theta), \tag{2.3b}$$
$$V(r, \theta, z, 0) = V_t(r, \theta, z, 0) = 0, \tag{2.3c}$$
$$V(r_0(\theta), \theta, z, t) = M(\theta, z, t) e^{-i\omega t}. \tag{2.3d}$$

Conditions (2.3a, b) correspond to an earth characterized by a variable surface impedance $\varepsilon^{-1} X(r, \theta)$ and a perfectly conducting ionosphere characterized by a variable height $\varepsilon h(r, \theta)$. Both X and h are assumed to be $O(1)$.

The assumption that the surface impedance is $O(\varepsilon^{-1})$ is physically motivated. Cases with $X = O(\varepsilon)$ or $X = O(\varepsilon^{-1})$ may occur, and are less general and easier to handle. Other linear boundary conditions could be assumed as the physical problem at hand dictates. Conditions (2.3c, d) assume a quiescent guide at $t=0$, and a given modulated signal with carrier frequency ω and slowly varying amplitude M at $r = r_0(\theta)$, which characterizes the source region. More detailed models could lead to equations that are more general than (2.1), such as equations with variable coefficients and containing first order derivatives [1], higher order equations [2], or systems of equation coupled by the boundary conditions [3]. They can be treated very much in the same way.

Our first step is to scale the z variable as follows:

$$z = \varepsilon \zeta; \quad V(r, \theta, z, t) = v(r, \theta, \zeta, t; \varepsilon). \tag{2.4}$$

This changes (2.1), (2.3) to

$$c^2(\Box v + \varepsilon^{-2} v_{\zeta\zeta}) = 0, \tag{2.5a}$$
$$v = 0, \quad \text{at} \quad \zeta = h(r, \theta), \tag{2.5b}$$
$$v_\zeta + X v = 0, \quad \text{at} \quad \zeta = 0, \tag{2.5c}$$
$$v = v_t = 0, \quad \text{at} \quad t = 0, \tag{2.5d}$$
$$v(r_0, \theta, \zeta, t; \varepsilon) = \mu(\theta, \zeta, t) e^{-i\omega t}, \tag{2.5e}$$

where

$$\Box \equiv \Delta - c^2 \partial_{tt}, \tag{2.6a}$$

$$\Delta \equiv \nabla \cdot \nabla \equiv \partial_{rr} + \frac{1}{r} \partial_r + \frac{1}{r^2} \partial_{\theta\theta}. \tag{2.6b}$$

We shall seek solutions to (2.5), for

$$t > 0, \quad r_0 \leqq r < \infty, \quad 0 \leqq \theta \leqq 2\pi, \quad 0 \leqq \zeta \leqq h, \tag{2.7}$$

as asymptotic expansions in the small parameter ε.

3. Asymptotic solution

We make the following basic assumptions about the form of the solution:

$$v(r, \theta, \zeta, t; \varepsilon) = w(r, \theta, \zeta, t, \xi; \varepsilon), \tag{3.1a}$$

where

$$\xi = \varepsilon^{-1} \phi(r, \theta, t), \tag{3.1b}$$

with $\phi = O(1)$ to be determined. By (3.1a) we assume the solution to be a function of the phase ξ, which henceforth we treat as an additional independent variable, and by (3.1b) we assume the phase ξ to be a fast varying (i.e., $O(\varepsilon^{-1})$) function of r, θ and t. Further we assume that there exists an asymptotic expansion

$$w \sim \sum_{n=0}^{\infty} w^{(n)}(r, \theta, \zeta, t, \xi)\varepsilon^n . \tag{3.2}$$

From (3.1a) we get

$$v_\zeta = w_\zeta , \quad v_r = w_r + \frac{\phi_r}{\varepsilon} w_\xi , \quad v_\theta = w_\theta + \frac{\phi_\theta}{\varepsilon} w_\xi , \quad v_t = w_t + \frac{\phi_t}{\varepsilon} w_\xi ,$$

and analogous expressions for higher derivatives.

In terms of the new variables, (2.5a) becomes

$$\varepsilon^{-2}\{c^2 w_{\zeta\zeta} + [c^2(\nabla\phi)^2 - \phi_t^2] w_{\xi\xi}\} +$$
$$+\varepsilon^{-1}\left[2\left(c^2\nabla\phi\cdot\nabla - \phi_t \frac{\partial}{\partial t}\right) + c^2 \Delta\phi\right] w_\xi + \Box w = 0 . \tag{3.3}$$

Substituting (3.2) in (3.3) yields

$$L_0 w^{(0)} + \varepsilon(L_0 w^{(1)} + L_1 w^{(0)}) + \varepsilon^2(L_0 w^{(2)} + L_1 w^{(1)} + \Box w^{(0)}) +$$
$$+ \ldots + \varepsilon^n(1_0 w^{(n)} + L_1 w^{(n-1)} + \Box w^{(n-2)}) + \ldots = 0 \tag{3.4}$$

from which we get the recursive system of equations

$$L_0 w^{(n)} = -L_1 w^{(n-1)} - \Box w^{(n-2)} , \quad n = 0, 1, 2, \ldots , \tag{3.5}$$

with $w^{(-1)} \equiv w^{(-2)} \equiv 0$, and

$$L_0 \equiv c^2\{\partial_{\zeta\zeta} + [(\nabla\phi)^2 - c^{-2}\phi_t^2]\partial_{\xi\xi}\} , \tag{3.6a}$$
$$L_1 \equiv 2c^2[(\nabla\phi\cdot\nabla - c^{-2}\phi_t\partial_t) + \tfrac{1}{2}\Delta\phi]\partial_\xi . \tag{3.6b}$$

In (3.3), (3.6), ∇ is the gradient operator $(\partial_r, r^{-1}\partial_\theta)$. The boundary and initial conditions for $w^{(n)}$ become

$$w^{(n)} = 0 , \quad \text{at} \quad \zeta = h , \tag{3.7a}$$
$$w_\zeta^{(n)} + X w^{(n)} = 0 , \quad \text{at} \quad \zeta = 0 , \tag{3.7b}$$
$$w^{(n)} = w_t^{(n)} = 0 , \quad \text{at} \quad t = 0 , \tag{3.7c}$$
$$w^{(0)} = \mu(\theta, \zeta, t)\mathrm{e}^{-i\omega t} , \quad \text{at} \quad r = r_0(\theta) , \tag{3.7d}$$
$$w^{(n)} = 0 , \quad \text{at} \quad r = r_0(\theta) , \quad \text{for} \quad n \geqq 1 . \tag{3.7e}$$

Equations (3.5), (3.7) for $n=0$ can be solved by separation of variables in ζ and ξ. The expression

$$w^{(0)} = \sum_{m=1}^{\infty} [A_m^{(0)}(r, \theta, t)\mathrm{e}^{i\gamma_m\xi} + B_m^{(0)}(r, \theta, t)\mathrm{e}^{-i\gamma_m\xi}] \sin[\beta_m(h-\zeta)] , \tag{3.8}$$

satisfies (3.5), (3.7a) and (3.7b), provided that

$$\gamma_m^2 = -\beta_m^2[(\nabla\phi)^2 - c^{-2}\phi_t^2]^{-1} \tag{3.9}$$

and

$$-\beta_m \cos\beta_m h + X \sin\beta_m h = 0 \Rightarrow \beta_m h \cot\beta_m h = Xh . \tag{3.10}$$

Matching (3.8) to (3.7d) and (3.7c) yields

$$\gamma_m(r_0(\theta), \theta, t)\xi(r_0(\theta), \theta, t) = -\omega t , \tag{3.11a}$$
$$\sum_m A_m^{(0)}(r_0(\theta), \theta, t)\sin[\beta_m(h-\zeta)] = \mu(\theta, \zeta, t) , \tag{3.11b}$$

$$B_m^{(0)}(r_0(\theta), \theta, t) = 0\,, \tag{3.11c}$$

$$A_m^{(0)}(r, \theta, 0) = B_m^{(0)}(r, \theta, 0) = 0\,. \tag{3.11d}$$

It will be seen later that (3.11c, d) imply

$$B_m^{(0)} \equiv 0\,. \tag{3.12}$$

Eq. (3.10) is independent of t. Since X and h are given, it can be solved for any (r, θ), yielding a set of values

$$\beta_m(r, \theta), \qquad m = 1, 2, 3, \ldots \tag{3.13}$$

The set of "modal eigenfunctions" appearing in (3.8)

$$f_m(\zeta) = \sin \beta_m(h-\zeta)\,, \tag{3.14a}$$

are solutions of the Sturm Liouville system

$$f_m'' + \beta^2 f_m = 0\,, \quad \text{with} \quad f_m(h) = 0\,,\ f_m'(0) + X f_m(0) = 0\,, \tag{3.14b}$$

and thus form a complete orthogonal set. Thus (3.11b) is a Fourier expansion of $\mu(\theta, \zeta, t)$ for $0 \leq \zeta \leq h$, and the Fourier coefficients are

$$A_m^{(0)}(r_0(\theta), \theta, t) = \langle \mu, f_m \rangle \equiv \frac{2}{\beta_m h} \int_0^h \mu(\theta, \zeta, t) \sin \beta_m(h-\zeta) d\zeta\,. \tag{3.15}$$

Eq. (3.15) gives the values of the modal amplitudes $A_m^{(0)}$ on the initial curve $r_0(\theta)$, in terms of the given function μ.

We now return to (3.5), (3.7) for $n=1$. Using (3.8) and noting that ξ is treated as an independent variable, we get

$$L_0 w^{(1)} = -L_1 w^{(0)} = \sum_m \left[\xi F_m(r, \theta, t, \zeta, \xi) + G_m(r, \theta, t, \zeta, \xi)\right] e^{i\gamma_m \xi} \tag{3.16a}$$

where

$$F_m = -2c^2 \gamma_m (\nabla\phi \cdot \nabla\gamma_m - c^{-2} \phi_t \gamma_{m,t}) A_m^{(0)} \sin \beta_m(h-\zeta)\,, \tag{3.16b}$$

$$G_m = 2ic^2 \left\{ \left[\gamma_m (\nabla\phi \cdot \nabla A_m^{(0)} - c^{-2} \phi_t A_{m,t}^{(0)}) + \left(\nabla\phi \cdot \nabla\gamma_m - c^{-2} \phi_t \gamma_{m,t} + \right.\right.\right.$$
$$\left.\left.\left. + \frac{\gamma_m}{2} \Delta\phi \right) A_m^{(0)} \right] \sin \beta_m(h-\zeta) + A_m^{(0)} [\nabla\phi \cdot \nabla\beta_m(h-\zeta) - \beta_m \gamma_m \nabla\phi \cdot \nabla h] \cos \beta_m(h-\zeta) \right\}. \tag{3.16c}$$

(Another expression, proportional to $\exp(-i\gamma_m \xi)$ has been deleted for reasons that will be elaborated later.) The boundary conditions are given by (3.7). It is obvious that unless $F_m = 0$, $w^{(1)}$ will be proportional to ξ^2, i.e., $O(\varepsilon^{-2})$. In other words: $\Sigma \xi F_m \exp(i\gamma_m \xi)$ is a secular term. This yields

$$\nabla\phi \cdot \nabla\gamma_m - c^{-2} \phi_t \gamma_{m,t} = 0\,, \tag{3.17}$$

which together with (3.9) constitutes a pair of coupled first order partial differential equations for γ_m and ϕ. Fortunately, we do not need to know γ_m and ϕ separately. Our solution (3.8) depends only on the product $\gamma_m \xi$, thus we may define (see (3.1b))

$$\gamma_m \xi \equiv \xi_m \Rightarrow \gamma_m \phi = \phi_m(r, \theta)\,. \tag{3.18}$$

We can pick any γ_m that satisfies (3.17). In particular, we can choose $\gamma_m = \text{constant} \neq 0$, which is a solution of (3.17). Then (3.9) becomes an equation for ϕ_m only:

$$(\nabla\phi_m)^2 - c^{-2} \phi_{m,t}^2 + \beta_m^2 = 0\,. \tag{3.19}$$

This is the eiconal equation for the modal phase functions ξ_m. Because β_m and c are independent of t, the solution must be of the form

$$\phi_m(r, \theta, t) = \tau_m(r, \theta) - \lambda_m t\,. \tag{3.20}$$

Substituting (3.20) in (3.19) yields

$$(\nabla\tau_m)^2 = \lambda_m^2 c^{-2} - \beta_m^2 \equiv N_m^2(r, \theta) \tag{3.21}$$

where β_m is determined by (3.10). N_m can be regarded as a "modal refractive index". It depends on X and h via β_m. Matching (3.20) to (3.11a) yields

$$\tau_m(r_0(\theta), \theta) = 0\,, \tag{3.22a}$$

$$\lambda_m = \lambda = \varepsilon\omega\,. \tag{3.22b}$$

Eq. (3.21), with the initial condition (3.22a) can be solved by characteristics or ray methods [6], [7], [8]:

$$\tau_m(r, \theta) = \int_0^s N_m[r, (s), \theta(s)]\, ds\,, \tag{3.23a}$$

with N_m defined by (3.21). The integral is taken over a ray that emanates from a point on the initial curve $r_0(\theta)$, and s is an arclength parameter along the path. The rays are determined by the ordinary differential equations [8]

$$\frac{d}{ds}\left[N_m(\boldsymbol{r}(s))\frac{d\boldsymbol{r}}{ds}\right] = \nabla N_m\,, \tag{3.23b}$$

where $\boldsymbol{r}$ is the radius vector of a point (r, θ). We also note, for future use, that the operator $\nabla\phi_m \cdot \nabla$ is simply related to the directional derivative along a ray [8]:

$$\nabla\phi_m \cdot \nabla = N_m \frac{\partial}{\partial s}\,. \tag{3.24}$$

We now consider ϕ_m and β_m as known, and use (3.17)–(3.24) to rewrite (3.16a) in the form

$$L_0 w^{(1)} = 2ic^2 \sum_m \left\{\left[N_m \frac{\partial A_m^{(0)}}{\partial s} + \frac{\lambda}{c^2}\frac{\partial A_m^{(0)}}{\partial t} + \tfrac{1}{2}\Delta\phi_m A_m^{(0)}\right] \sin\beta_m(h-\zeta) + \right.$$
$$\left. + N_m[\beta_m'(h-\zeta) - \beta_m h']\, A_m^{(0)} \cos\beta_m(h-\zeta)\right\} e^{i\xi_m}\,, \tag{3.25a}$$

where

$$\beta_m' = \frac{d\beta_m}{ds}\,, \qquad h' = \frac{dh}{ds}\,, \tag{3.25b}$$

since β_m and h do not depend on t.

We can solve (3.25) as follows. We let

$$w^{(1)} = p^{(1)} + q^{(1)} + r^{(1)} \tag{3.26a}$$

where $p^{(1)}$, $q^{(1)}$ and $r^{(1)}$ satisfy the boundary conditions (3.7a, b) and $p^{(1)}$ is a solution of the homogeneous problem

$$L_0 p^{(1)} = 0\,. \tag{3.26b}$$

Being a solution of the homogeneous problem, $p^{(1)}$ must have the same form as $w^{(0)}$, namely,

$$p^{(1)} = \sum_m A_m^{(1)}(r, \theta, t) \sin[\beta_m(h-\zeta)]\, e^{i\xi_m}\,. \tag{3.27}$$

We can make a special choice of $q^{(1)}$ that will eliminate the terms in (3.25) which are proportional to $\cos\beta_m(h-\zeta)$. This will result in obtaining a set of uncoupled equations for the determination of $A_m^{(0)}$.

If we choose

$$q^{(1)} = \sum_m [P_m(r,\theta,t)(h-\zeta)^2 \sin\beta_m(h-\zeta) + Q_m(r,\theta,t)(h-\zeta)\cdot$$
$$\sin\beta_m(h-\zeta) + R_m(r,\theta,t)(h-\zeta)\cos\beta_m(h-\zeta)]\, e^{i\xi_m}, \tag{3.28a}$$

it is obvious that (3.7a) is satisfied. A simple calculation, using (3.10), shows that (3.7b) will be satisfied if

$$2hP_m + Q_m = \left[(\beta_m + X)h + \frac{X}{\beta_m}(Xh-1)\right] R_m \equiv \Gamma_m R_m, \tag{3.28b}$$

where $\Gamma_m(r,\theta)$ is the known expression in the square brackets. From (3.25), (3.26) we get

$$L_0 w^{(1)} = L_0 q^{(1)} + L_0 r^{(1)} = -L_1 w^{(0)}. \tag{3.29a}$$

Using (3.19) it is not hard to show that

$$L_0 q^{(1)} = 2c^2 \sum_m [2\beta_m P_m (h-\zeta)\cos\beta_m(h-\zeta) +$$
$$+ \beta_m Q_m \cos\beta_m(h-\zeta) + (P_m - 2\beta_m R_m)\sin\beta_m(h-\zeta)]\, e^{i\xi_m}. \tag{3.29b}$$

Thus, if we choose

$$P_m = \frac{i}{2}\beta_m'/\beta_m A_m^{(0)}, \tag{3.30a}$$

$$Q_m = ih' A_m^{(0)}, \tag{3.30b}$$

Eq. (3.29a) becomes

$$L_0 r^{(1)} = 2ic^2 \sum N_m [K_m A_m^{(0)}] \sin\beta_m(h-\zeta)\, e^{i\xi_m}, \tag{3.31a}$$

where K_m is the linear operator

$$K_m \equiv \frac{\partial}{\partial s} + \frac{\lambda}{c^2 N_m}\frac{\partial}{\partial t} + \tfrac{1}{2}\left[\frac{\Delta\phi_m}{N_m} + \frac{2\beta_m h}{\Gamma_m}\frac{h'}{h} - \left(1 - \frac{2\beta_m h}{\Gamma_m}\right)\frac{\beta_m'}{\beta_m}\right]. \tag{3.31b}$$

The right hand side of (3.31a) is proportional to a solution of the homogeneous problem, and is thus a secular term. If $w^{(1)}$ is to be $O(1)$, i.e., not proportional to ξ_m, we must set

$$K_m A_m^{(0)} \equiv A_{m,s}^{(0)} + \frac{\lambda}{c^2 N_m} A_{m,t}^{(0)} + D_m A_m^{(0)} = 0, \tag{3.32}$$

where the subscripts s, t mean differentiations with respect to s, t. This is a linear equation which, together with the initial conditions (3.11d) and (3.15), determines $A_m^{(0)}$ uniquely. It is also seen that if we would have retained the second term in (3.8) all the way, we would have obtained (3.12). This is because B_m is the solution of a linear equation like (3.32) with zero initial conditions.

Since the coefficients of (3.32) are independent of t, an explicit solution can be obtained, determining $w^{(0)}$ completely (see (4.6) in next section). We can now, without loss of generality, set $r^{(1)} = 0$, and then

$$w^{(1)} = p^{(1)} + q^{(1)} \tag{3.33}$$

with $q^{(1)}$ completely determined by (3.28), (3.30), and $p^{(1)}$ given by (3.27). The coefficients $A_m^{(1)}$, $B_m^{(1)}$ in (3.27) can be determined in the same way by solving (3.5) for $n=2$.

Solving (3.5) (with the conditions (3.7)) for $n \geqq 2$ follows the same procedure. The coefficients $A_m^{(n)}$, $B_m^{(n)}$ will be solutions of linear equations like (3.32), with non-homogeneous terms which depend on $A_m^{(n-k)}$, $B_m^{(n-k)}$. Thus it is possible (in principle) to calculate any desired number of terms in (3.2).

4. Interpretation of the results

The zero order approximation of the problem posed in sec. 2 is

$$v \sim w^{(0)} + O(\varepsilon) \tag{4.1}$$

where $w^{(0)}$ is given by (3.8) as a set of uncoupled modes, each with its own phase and amplitude. The modal phase functions ξ_m are given by (3.1b), (3.18), (3.20), (3.22) and (3.23a). From (3.20) we get for ϕ_m=constant

$$\nabla\tau_m \cdot \frac{dr}{dt} - \lambda_m = 0 . \tag{4.2}$$

We define as the modal phase velocity the magnitude of dr/dt when dr is parallel to $\nabla\tau_m$, i.e., in the ray direction. Thus, from (4.2) and (3.21) the modal phase velocity is

$$C_{pm} = \frac{\lambda}{N_m} = c\left[1 - \left(\frac{c\beta_m}{\lambda}\right)^2\right]^{-\frac{1}{2}} . \tag{4.3}$$

With β_m determined by h and X via (3.10) and λ determined by the carrier frequency via (3.22b), the local phase velocity is determined. Due to (3.23a) we can write the phase function in the form

$$\phi_m = \lambda\left[\int_0^s \frac{ds}{C_{pm}} - t\right] . \tag{4.4}$$

It is convenient to define a group velocity by

$$C_{gm} C_{pm} = c^2 \Rightarrow C_{gm} = c^2 N_m \lambda^{-1} . \tag{4.5}$$

In terms of (4.5), the solution of (3.32) subject to (3.15) can be found by means of the Laplace transform, and is given below. (Note that $s=0$ on $r=r_0(\theta)$).

$$A_m^{(0)}(s, t) = A_m^{(0)}\left(0, t - \int_0^s \frac{ds'}{C_{gm}(s')}\right) \exp\left[-\int_0^s D_m(s')\, ds'\right] . \tag{4.6}$$

The modal amplitudes propagate along the rays at the group velocity, while undergoing a change of shape due to changes in β_m and h and due to geometrical spreading.

We see that the three-dimensional waveguide propagation problem has been reduced to a two-dimensional geometrical optics problem for each propagating mode, with each mode having its own equivalent refractive index. Coupling between the modes is detected when we look at the next approximation, namely

$$v \sim w^{(0)} + \varepsilon w^{(1)} + O(\varepsilon^2) . \tag{4.7}$$

From (3.27), (3.28), (3.30) and (3.33) we see that $w^{(1)}$ consists of two parts: $p^{(1)}$ is a set of uncoupled modes which are an $O(\varepsilon)$ correction of $w^{(0)}$. The information on mode coupling is contained in $q^{(1)}$. We may write (3.28a) as

$$q^{(1)} = \sum_m T_m(r, \theta, \zeta, t)\, e^{i\xi_m} , \tag{4.8}$$

where T_m satisfied conditions (3.7a, b), and expand T_m in a Fourier series for $0 \leqq \zeta \leqq h$

$$T_m(r, \theta, \zeta, t) = \sum_k J_k^m(r, \theta, t) \sin \beta_k(h-\zeta) , \tag{4.9a}$$

with

$$J_k^m(r, \theta, t) = \langle T_m, f_k \rangle , \tag{4.9b}$$

where f_m is given by (3.14a), the inner product $\langle\ ,\ \rangle$ is defined by (3.15) and is given explicitly by (4.10b). Thus $q^{(1)}$ becomes

$$q^{(1)}(r, \theta, \zeta, t) = \sum_m \sum_k J_k^m(r, \theta, t) \sin \beta_k(h-\zeta) \quad e^{i\xi_m} \tag{4.10a}$$

where

$$J_k^m = \frac{2}{\beta_k h} \int_0^h \sin \beta_k(h-\zeta)[P_m(h-\zeta) \sin \beta_m(h-\zeta) +$$
$$+ Q_m \sin \beta_m(h-\zeta) + R_m \cos \beta_m(h-\zeta)](h-\zeta)\, d\zeta . \tag{4.10b}$$

P_m, Q_m, R_m are given by (3.30) and (3.28) respectively and are proportional to $A_m^{(0)}$. J_k^m is the coupling coefficient between the mth mode in $w^{(0)}$ and the kth mode in $w^{(1)}$.

Since β_m is an increasing sequence, it is clear from (3.21) that there can be at most a finite number of propagating modes at any point. By solving the ray equations (3.23b) for each mode, we may find caustics (envelopes of ray families) and foci in the r, θ plane. We know that our expansion does not hold in the vicinity of such curves or points, and a uniform expansion is necessary [9]. However, knowledge of the locations of such curves or points is important by itself, since they indicate regions of field enhancement and shadow regions for each mode.

We should note that this method, which is related to the multiple-scale method [10], is rather versatile. Many generalizations and refinements are possible. It has also been used in studying some nonlinear problems (though with less success, due to the great difficulties that are inherent to such problems). Its success requires a good choice of scaling (in our case (2.4) and (3.1)), which is a non-trivial problem [11]. More refined results, such as expansions that are uniformly valid across caustics or turning points can be obtained in this way too. This would require a choice of a phase function different from our (3.1), but then again no a priori "Ansatz" is necessary.

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